GRAPHIC ALGEBRA

FOR SECONDARY SCHOOLS

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PREFACE

This pamphlet is intended to meet the growing demand in this country for graphical methods and illustrations in connection with elementary Algebra in the secondary schools. A recent committee of the American Mathematical Society, in formulating standard college entrance requirements in Algebra, uses this language: "The use of graphical methods and illustrations, particularly in connection with the solution of equations, is also expected."

This little book is the author's interpretation of the committee's recommendation both as to kind and amount of such graphical work. It may be used to supplement any text-book in Algebra.

In plotting the graphs of equations the pupil should use "squared paper" similar to the sample sheet in the back of the pamphlet.

The teacher or pupil wishing further information on the subject may consult any standard work on Analytic Geometry.

H. B. NEWSON

LAWRENCE, KANSAS
January, 1905

GRAPHIC ALGEBRA

DEFINITIONS

Axes. Two lines intersecting at right angles, one horizontal and the other vertical as in Fig. 1, are called the axes of coördinates. The horizontal line is called the X-axis, or the axis of abscissas; the vertical line is called the Y-axis, or the axis of ordinates. O, the point of intersection of the axes, is called the origin.

Quadrants. The axes of coördinates divide the plane into four equal parts called quadrants. These are numbered from I to IV, as shown in Fig. 1.

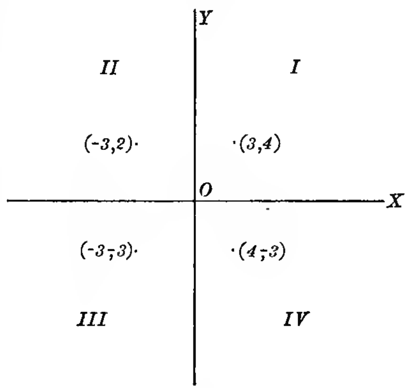

Fig. 1

Coördinates. A point is located in the plane when its perpendicular distance and direction from each of the axes are known. The distance of a point from OY is called the x-distance or abscissa; its distance from OX is called the y-distance or ordinate; the two distances are

called the coördinates of the point. The position of a point is indicated by the symbol (x, y); thus $(3, 4)$ denotes a point whose x-distance is 3 and whose y-distance is 4. (It should be noted in this symbol that the x-distance is always written first and the y-distance second.)

Signs of the Coördinates. Distances to the right of the Y-axis are called positive; those to the left, negative. Distances above the X-axis are positive; those below, negative. Thus a point in the first quadrant has both of its coördinates positive; one in the second quadrant has the x-coördinate negative and the y-coördinate positive; one in the third quadrant has both coördinates negative; one in the fourth quadrant has the x-coördinate positive and the y-coördinate negative. These are shown in Fig. 1.

Plotting. The process of locating a point by means of its coördinates is called plotting the point. To locate the point $(3, 4)$, we measure from the origin to the right on OX three units, and then from this point we measure up four units parallel to OY; the point thus reached is the point $(3, 4)$. The point $(-3, 2)$ is in the second quadrant, three units to the left of OY and two units above OX. The point $(-3, -3)$ is in the third quadrant, three units to the left of OY and three units below OX. The point $(4, -3)$ is in the fourth quadrant, four units to the right of OY and three units below OX.

Every point in the plane has a pair of coördinates, and, conversely, to every pair of numbers there corresponds some point whose coördinates are these numbers. These numbers may be integers, fractions, or irrational numbers; but they must be real for the reason that there is no point one or both of whose coördinates are imaginary.

EXERCISE I

1. Plot the following points.

(*a*) $(3, 5)$.
(*b*) $(3, -5)$
(*c*) $(-4, -2)$.
(*d*) $(-3, -5)$.
(*e*) $(-3, 2)$.
(*f*) $(7, 3)$.
(*g*) $(3, 7)$.
(*h*) $(3, -7)$.
(*i*) $(7, -3)$.
(*j*) $(-3, -7)$.
(*k*) $(2\frac{1}{2}, 10)$.
(*l*) $(4\frac{2}{3}, -\frac{2}{5})$.
(*m*) $(-\frac{1}{2}, 1\frac{1}{2})$.
(*n*) $(3.6, -4\frac{3}{5})$.
(*o*) $(5\frac{1}{2}, 3\frac{1}{3})$.
(*p*) $(3.3, 2)$.
(*q*) $(-.6, 2.4)$.
(*r*) $(1, \sqrt{2})$.
(*s*) $(-\frac{3}{10}, .3)$.
(*t*) $(-\sqrt{5}, \frac{1}{2}\sqrt{3})$.

2. Where are the points $(0, 4)$, $(-4, 0)$?

3. What are the coördinates of the origin?

4. Show that all points whose ordinates are 0 are on the X-axis, and all whose abscissas are 0 are on the Y-axis.

THE LINEAR EQUATION AND ITS GRAPH

We now go on to show how the above method of plotting points may be used to illustrate some simple topics in Algebra. Let us take a first-degree or linear equation in two variables, as $3x + 4y = 12$. Such an equation is called indeterminate for the reason that we can find an unlimited number of pairs of values of x and y which satisfy it. (A pair of values of x and y are said to satisfy an equation if, when substituted for x and y in the equation, they make the two sides of the equation numerically equal.) The following are pairs of values which satisfy this equation.

$x = -3$	$y = 5\frac{1}{4}$	$x = 2$	$y = 1\frac{1}{2}$
$x = -2$	$y = 4\frac{1}{2}$	$x = 3$	$y = \frac{3}{4}$
$x = -1$	$y = 3\frac{3}{4}$	$x = 4$	$y = 0$
$x = 0$	$y = 3$	$x = 5$	$y = -\frac{3}{4}$
$x = 1$	$y = 2\frac{1}{4}$	etc.	etc.

These and other pairs of values are found by giving to one of the variables, as x, any value we choose and then solving the resulting equation for y. Thus let $x = 10$, whence $3(10) + 4y = 12$, or $y = -4\frac{1}{2}$; $x = 10$ and $y = -4\frac{1}{2}$ are a pair of values satisfying the equation $3x + 4y = 12$.

These pairs of values may now be taken to be the coördinates of a series of points. Let us plot the points $(-3, 5\frac{1}{4})$, $(0, 3)$, $(3, \frac{3}{4})$, $(4, 0)$ $(10, -4\frac{1}{2})$, etc. It is seen at once that these points lie on a straight line, Fig. 2, called the graph of the equation. On the other hand, if from any point on this line perpendiculars be let fall to the axes and these perpendiculars be measured, the numbers thus obtained will satisfy the equation.

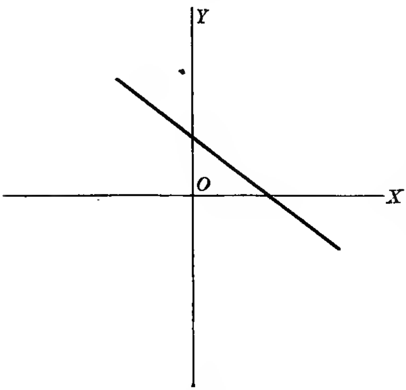

FIG. 2

It may readily be shown by trial that every equation of the first degree has a straight line as a graph. An equation of the first degree is therefore called a linear equation.

A Shorter Method. Knowing that the graph of a linear equation is a straight line, the easiest way to construct the line is to locate the two points where it crosses the axes. The point where the graph crosses the X-axis is found by putting $y = 0$ in the equation and solving for x. The point where it crosses the Y-axis is found by putting $x = 0$ in the equation and solving for y.

For example, let us construct the graph of the equation

$$4x + 6y = 12.$$

Put $y = 0$, then $x = 3$;

put $x = 0$, then $y = 2$.

Plot the points (3, 0) and (0, 2); the line determined by these two points is the required graph.

This method fails when the line passes through the origin; in this case one point must be located not on either axis; the line through this point and the origin is the required line.

EXAMPLE. $2x - 5y = 0$. If $x = 0$, $y = 0$; if $x = 5$, $y = 2$. The required graph is the line, Fig. 3, through the points (5, 2) and (0, 0).

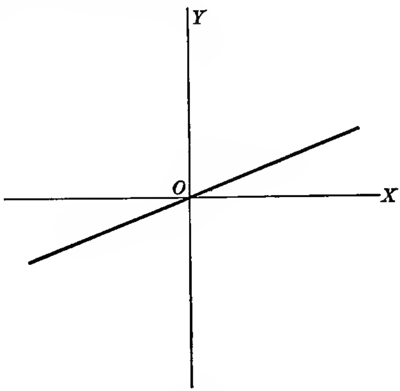

FIG. 3

EXERCISE II

1. Construct the graphs of the following linear equations.

(*a*) $3x + 4y = 7.$

(*b*) $x - 6y = -3.$

(*c*) $3x - 4y = 6.$

(*d*) $3y = 2x - 10.$

(*e*) $\frac{x-5}{2} = y + 1.$

(*f*) $6x - y = 0.$

(*g*) $\frac{2x}{3} = -\frac{y}{2}.$

(*h*) $2x = 10 + 3y.$

(*i*) $y = -4x + 6.$

(*j*) $y = x - 4.$

2. What is the graph of the equation $x = 0$? of $y = 0$? of $y = 4$? of $x = -3$?

3. What are the equations of the X- and Y-axes respectively?

4. What are the equations of the two lines bisecting the angles between the two axes?

SIMULTANEOUS LINEAR EQUATIONS

Intersecting Lines. Let

$$\begin{cases} 5x + 4y = 22, \\ 3x + y = 9, \end{cases}$$

be a pair of simultaneous linear equations. Draw the graphs of these two equations, using the same set of axes, Fig. 4.

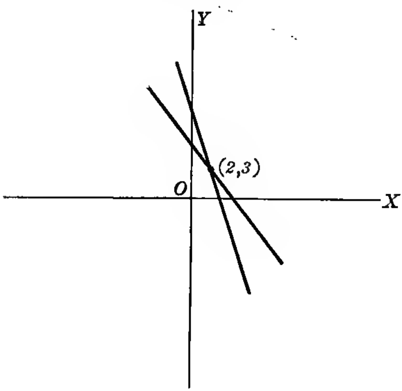

FIG. 4

By measurement we find that the point of intersection of these two lines has the coördinates (2, 3). If we solve the two equations for x and y, we find the values $x = 2$, $y = 3$. Thus we see that the values of x and y, which we get as the one solution of a pair of simultaneous equations, are the coördinates of the point of intersection of the graphs of the equations. This point is on both lines and its coördinates satisfy both equations.

Parallel Lines. If the two given equations are

$$\begin{cases} 2x + 3y = 6, \\ 4x + 6y = 21, \end{cases}$$

we meet with a difficulty when we try to solve for x and y. The reason is that we cannot eliminate x without at the same time eliminating y. If we construct the graphs of these two equations, we find that they are parallel lines, Fig. 5, and so do not intersect at all. This peculiarity of the graphs explains the algebraic difficulty. The lines have no point of intersection and hence the equations of the lines have no solution.

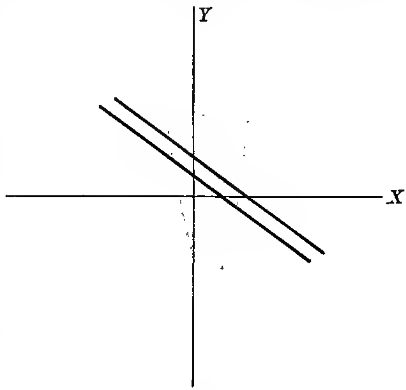

FIG. 5

EXERCISE III

Find both algebraically and graphically the points of intersection of the following pairs of lines.

1. $\begin{cases} 5x - 3y = 20, \\ 2x + 5y = 39. \end{cases}$

2. $\begin{cases} 5x + 2y = 39, \\ 2x - y = 3. \end{cases}$

3. $\begin{cases} 2x - y = 5, \\ x + 2y = 25. \end{cases}$

4. $\begin{cases} 4x + 2y = 13, \\ 2x + y = -9. \end{cases}$

5. $\begin{cases} x - 4 = 0, \\ y = 0. \end{cases}$

6. $\begin{cases} x + 3y = 22, \\ 2x - 4y = 4. \end{cases}$

7. $\begin{cases} 2x - 5y = 66, \\ 3x + 2y = 23. \end{cases}$

8. $\begin{cases} 3x + 2y = 0, \\ 5x - 6y = 0. \end{cases}$

9. $\begin{cases} x - y = 1, \\ 6x - 6y = 1. \end{cases}$

10. $\begin{cases} 2y + 3 = 0, \\ x + 7 = 0. \end{cases}$

THE QUADRATIC EQUATION

Graph of the Quadratic. The general form of a quadratic equation is $ax^2 + bx + c = 0$. In order to treat this equation graphically we must join to it the equation $y = 0$ and treat the pair as a simultaneous system.

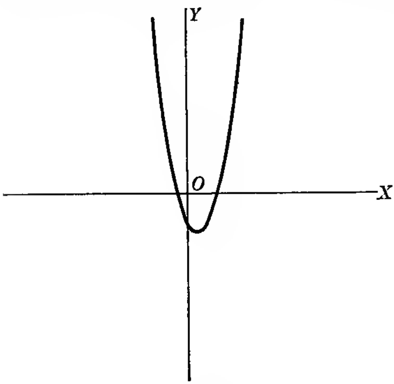

FIG. 6

Let us take the two equations

$$\begin{cases} x^2 - 2x - 3 = 0, \\ y = 0. \end{cases}$$

We may write them in the form

$$y = x^2 - 2x - 3.$$

Give to x a series of values and find the corresponding values of y. Thus

$x = -5$	$y = 32$	$x = 2$	$y = -3$
$x = -4$	$y = 21$	$x = 3$	$y = 0$
$x = -3$	$y = 12$	$x = 4$	$y = 5$
$x = -2$	$y = 5$	$x = 5$	$y = 12$
$x = -1$	$y = 0$	$x = 6$	$y = 21$
$x = 0$	$y = -3$	$x = 7$	$y = 32$
$x = 1$	$y = -4$	etc.	etc.

Plotting these points and connecting them by a smooth curve, we get the result shown in Fig. 6. The graph in this case is a curve and not a straight line. The graph of the

equation $y = 0$ is the X-axis. The curve cuts the X-axis in two points whose distances from the origin are -1 and 3.

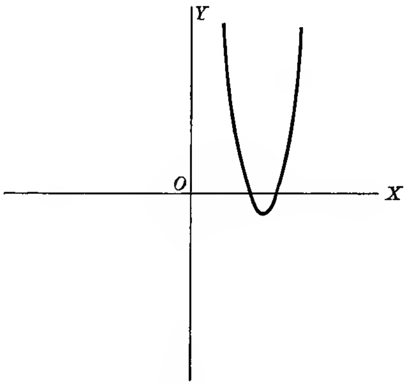

FIG. 7, a

If we solve the quadratic equation $x^2 - 2x - 3 = 0$, we get two roots, $x = -1$ and $x = 3$. Therefore the X-axis cuts the curve which is the graph of the equation $y = x^2 - 2x - 3$ in two points whose abscissas are the roots of the equation

$$x^2 - 2x - 3 = 0.$$

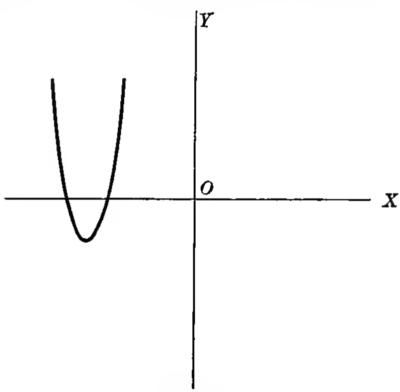

FIG. 7, b

Parabola. The curve which is the graph of the quadratic equation of the form

$$y = ax^2 + bx + c$$

is called a parabola.

It is the same curve, only inverted, as that described by a ball tossed into the air. The parabola obtained as the graph

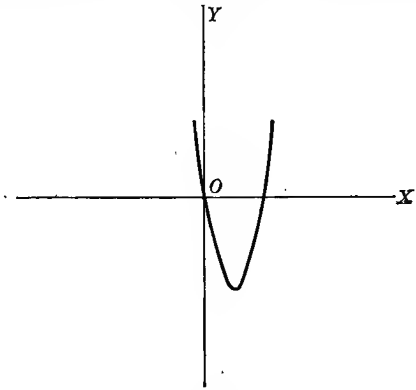

FIG. 7, c

of a quadratic equation has the property that it cuts the X-axis (or a line parallel to it) in two and only two points.

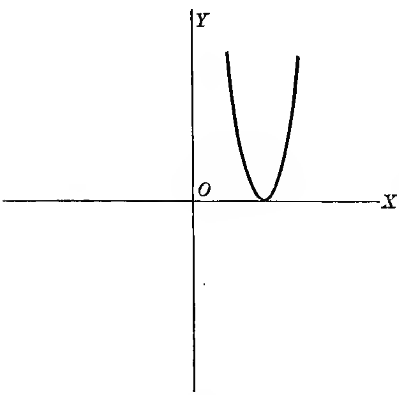

FIG. 7, d

Character of the Roots. Solving the quadratic $ax^2+bx+c=0$, we get

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

The roots are real when the expression

$$b^2-4ac>0;$$

they are equal when

$$b^2-4ac=0;$$

they are imaginary when

$$b^2-4ac<0.$$

The expression $b^2 - 4ac$ is called the discriminant of the quadratic.

The roots of the quadratic equation $ax^2 + bx + c = 0$ are the numbers which substituted for x in the expression $ax^2 + bx + c$ make it zero in value. They are also the x-coördinates of the points where the graph of the equation $y = ax^2 + bx + c$ crosses the X-axis.

The graph of the quadratic equation shows at once the character of the roots of the equation.

If the X-axis cuts the parabola in two points, the roots of the quadratic are real; this corresponds to the case $b^2 - 4ac > 0$. If these two points are both to the right of the origin, both roots are positive, Fig. 7, a; if the two points are both to the left of the origin, both roots are negative, Fig. 7, b; if these two points are on opposite sides of the origin, one root is positive and one negative, Fig. 6; if one of these points is at the origin, one root is zero, Fig. 7, c.

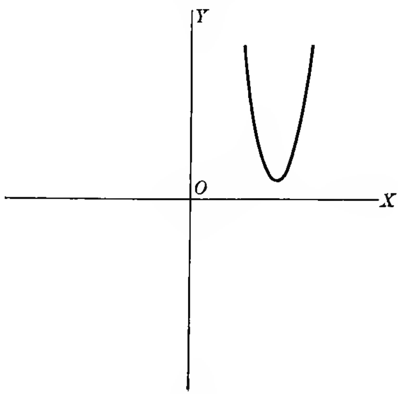

FIG. 7, e

If the X-axis touches the parabola (or as we say cuts it in two coincident points), the two roots are equal, Fig. 7, d; this corresponds to the case $b^2 - 4ac = 0$.

If the X-axis does not cut the parabola at all, the two roots of the equation are imaginary, Fig. 7, e; this corresponds to the case $b^2 - 4ac < 0$.

EXERCISE IV

Draw the graphs of the following equations and compare the roots found graphically with those found algebraically.

1. $x^2 - 9 = 0.$
2. $3x^2 - 5x = 2.$
3. $4x^2 - 23x + 30 = 0.$
4. $4x^2 + 4x + 3 = 0.$
5. $5x^2 - 7x = 0.$
6. $3x^2 + 5x + 9 = 0.$
7. $3x^2 + 8x = 0.$
8. $2x - 3 = 2/x.$
9. $x^2 + 8x + 16 = 0.$
10. $x^2 - 4x + 4 = 0.$

QUADRATIC EQUATIONS IN TWO VARIABLES

The graph of an equation of the second degree in two variables is always a curve. We have already treated one of these curves, the parabola. Our next task is to become acquainted with more of these curves.

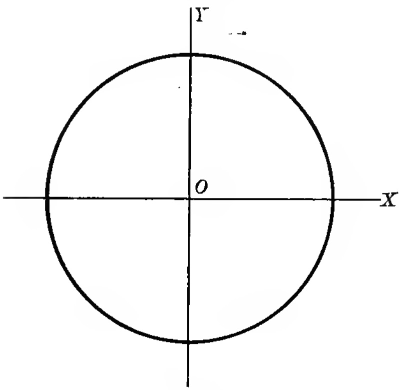

FIG. 8

The Circle. Plot the graph of the equation $x^2 + y^2 = 100.$

Solving for y, we get $y = \pm\sqrt{100 - x^2}.$

Giving to x a set of values, we find the corresponding values of y; thus

$x = 0$	$y = \pm 10$	$x = \pm 6$	$y = \pm 8$
$x = \pm 1$	$y = \pm 3\sqrt{11}$	$x = \pm 7$	$y = \pm\sqrt{51}$
$x = \pm 2$	$y = \pm 4\sqrt{6}$	$x = \pm 8$	$y = \pm 6$
$x = \pm 3$	$y = \pm\sqrt{91}$	$x = \pm 9$	$y = \pm\sqrt{19}$
$x = \pm 4$	$y = \pm 2\sqrt{21}$	$x = \pm 10$	$y = 0$
$x = \pm 5$	$y = \pm 5\sqrt{3}$	etc.	etc.

Plotting the points $(0, 10)$, $(0, -10)$, $(1, 3\sqrt{11})$, $(1, -3\sqrt{11})$, $(2, 4\sqrt{6})$, $(2, -4\sqrt{6})$, $(3, \sqrt{91})$, $(3, -\sqrt{91})$, $\cdots$, $(-6, +8)$, $(-6, -8)$, etc., we find that they all lie on a circle of radius 10, whose center is at the origin, Fig. 8.

The Ellipse. Plot the graph of the equation

$$4x^2 + 9y^2 = 144.$$

Solving for y, we get

$$y = \pm \tfrac{2}{3}\sqrt{36 - x^2}.$$

Give to x a set of values and find the corresponding values of y. We thus find the following set of points which belong to the graph of the equation: $(0, 4)$, $(0, -4)$, $(1, \frac{2}{3}\sqrt{35})$, $(1, -\frac{2}{3}\sqrt{35})$, $(-1, \frac{2}{3}\sqrt{35})$, $(-1, -\frac{2}{3}\sqrt{35})$, $(2, \frac{8}{3}\sqrt{2})$,

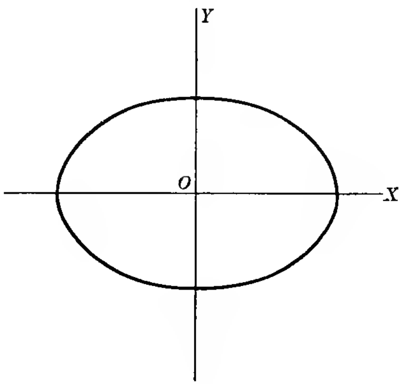

FIG. 9

$(2, -\frac{8}{3}\sqrt{2})$, $(-2, \frac{8}{3}\sqrt{2})$, $(-2, -\frac{8}{3}\sqrt{2})$, $(3, 2\sqrt{3})$, $(-3, 2\sqrt{3})$, etc. Plotting these points and drawing a smooth curve through them, we obtain the curve shown in Fig. 9. This curve is called an ellipse.

The Hyperbola. Plot the graph of the equation

$$9x^2 - 16y^2 = 144.$$

In the manner indicated above we find the following set of points: $(4, 0)$, $(-4, 0)$, $(5, \frac{9}{4})$, $(5, -\frac{9}{4})$, $(-5, \frac{9}{4})$, $(-5, -\frac{9}{4})$, $(6, \frac{3}{2}\sqrt{5})$, $(6, -\frac{3}{2}\sqrt{5})$, $(-6, \frac{3}{2}\sqrt{5})$, $(-6, -\frac{3}{2}\sqrt{5})$, etc.

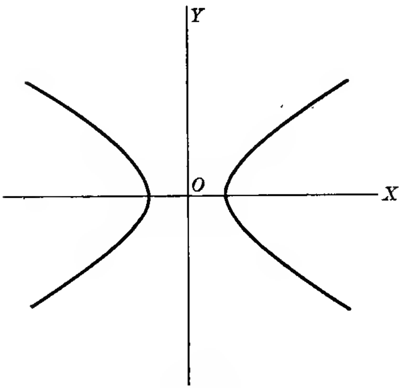

FIG. 10

Plotting these points and drawing a smooth curve through them, we obtain the curve shown in Fig. 10. This curve has two branches and is called an hyperbola. The two branches of the curve are exactly alike, but turned in opposite directions; thus the curve is symmetrical with respect to both the axes.

EXERCISE V

Draw the graphs of the following equations.

1. $x^2 + y^2 = 2y$.
2. $(x-2)^2 + (y-3)^2 = 4$.
3. $9x^2 + y^2 = 9$.
4. $2x^2 + 3xy + 2y^2 = 6$.
5. $x^2 - y^2 = 8$.
6. $4xy = 13$.

SIMULTANEOUS QUADRATIC EQUATIONS

The theory of simultaneous quadratic equations is beautifully illustrated and clarified by the use of graphical methods.

For these methods tell us at a glance the number of real solutions, the approximate values of the roots, and their proper pairing in order to constitute a solution.

Straight Line and Curve. Consider the pair of equations

$$\begin{cases} x - y = 4, \\ x^2 + y^2 = 40. \end{cases}$$

Solving the equations by one of the usual methods, we get as solutions two pairs of values of x and y, viz., $x=6$, $y=2$; $x=-2$, $y=-6$.

Now plot the graphs of the two equations on the same diagram, Fig. 11. The first equation being linear its graph is a straight line. The graph of the second equation turns out to be a circle of radius $2\sqrt{10}$. The line cuts the circle in two points, (6, 2) and (− 2, − 6). Thus we see that values of x and y which satisfy both equations are the coördinates of the points of intersection of the two graphs. We also see in this case that there are two and only two points of intersection and hence two and only two solutions of the equations.

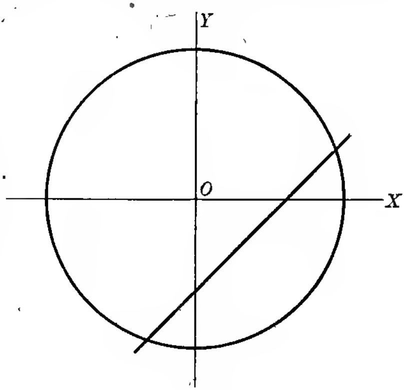

FIG. 11

Again take the pair of equations

$$\begin{cases} x + y = 7, \\ xy = 10. \end{cases}$$

Solving by the usual method, we get $x = 2$, $y = 5$; $x = 5$, $y = 2$. The graph of the first is a straight line and that

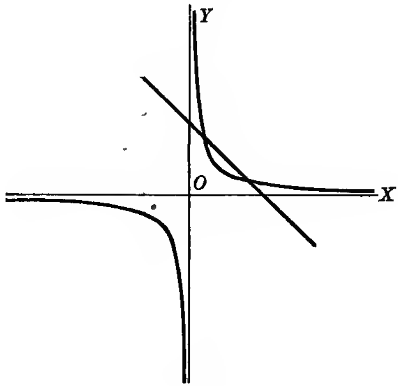

FIG. 12

of the second is an hyperbola, Fig. 12. This line cuts the hyperbola in the two points (2, 5) and (5, 2).

Consider also the pair of equations

$$\begin{cases} x + y = 8, \\ x^2 - xy + y^2 = 28. \end{cases}$$

The graph of the first is a straight line and that of the second an ellipse, Fig. 13. The two points of intersection of the line and ellipse are (2, 6) and (6, 2).

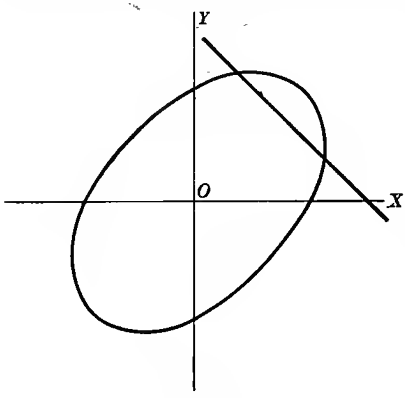

FIG. 13

The usual method of solution gives $x = 6$, $y = 2$; $x = 2$, $y = 6$.

Two Curves. Let the pair of equations be

$$\begin{cases} 4\,x^2 + 9\,y^2 = 36, \\ 4\,x^2 + 4\,y^2 = 25. \end{cases}$$

Plot the graph of each equation on the same set of axes. The first is an ellipse whose longest diameter is 6 and whose shortest diameter is 4; the second is a circle whose radius

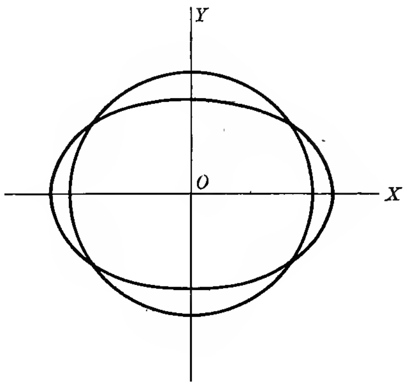

FIG. 14

is $\frac{5}{2}$, Fig. 14. The ellipse and circle cut in four points, viz., $(\frac{9}{10}\sqrt{5}, \frac{1}{5}\sqrt{55})$, $(\frac{9}{10}\sqrt{5}, -\frac{1}{5}\sqrt{55})$, $(-\frac{9}{10}\sqrt{5}, \frac{1}{5}\sqrt{55})$, $(-\frac{9}{10}\sqrt{5}, -\frac{1}{5}\sqrt{55})$.

Solving the pair of equations, we find four solutions corresponding to the four points of intersection of the graphs.

Take next the pair of equations

$$\begin{cases} y^2 = x, \\ xy = 10, \end{cases}$$

and plot their graphs, Fig. 15. The first curve is a parabola, the second an hyperbola. There is only one point of

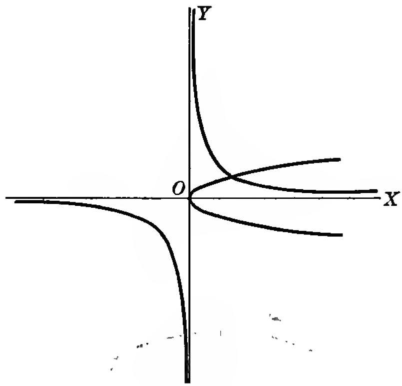

Fig. 15

intersection. Solving the equations, we find $x = \sqrt[3]{100}$ and $y = \sqrt[3]{10}$.

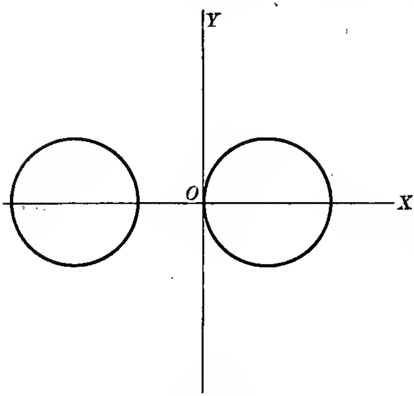

Fig. 16

Finally, consider the equations

$$\begin{cases} x^2 + y^2 - 2x = 0, \\ x^2 + y^2 + 4x + 3 = 0, \end{cases}$$

and plot their graphs on the same set of axes. The graphs are circles which do not intersect, Fig. 16. If we solve the equations by the usual method, we get $x = -\frac{1}{2}$,

$y = \pm \frac{1}{2}\sqrt{-5}$. These values of y are imaginary. There is no point one or both of whose coördinates is imaginary. This example illustrates the general principle that imaginary solutions of a pair of equations correspond to non-intersection of corresponding graphs.

EXERCISE VI

Solve both algebraically and graphically the following pairs of equations.

1. $\begin{cases} x + y = 12, \\ xy = 27. \end{cases}$

2. $\begin{cases} x - y = 9, \\ x^2 + y^2 = 45. \end{cases}$

3. $\begin{cases} 3x - y = 12, \\ x^2 - y^2 = 16. \end{cases}$

4. $\begin{cases} y - 3x = 1, \\ x^2 + xy = 33. \end{cases}$

5. $\begin{cases} 5x - 4y = 10, \\ 3x^2 - 4y^2 = 8. \end{cases}$

6. $\begin{cases} x^2 + 3y = 17, \\ 3x - y = 3. \end{cases}$

7. $\begin{cases} x^2 + xy = 24, \\ xy + y^2 = 40. \end{cases}$

8. $\begin{cases} x^2 + xy = 84, \\ x^2 - y^2 = 24. \end{cases}$

9. $\begin{cases} x^2 + y^2 - x - y = 78, \\ xy + x + y = 39. \end{cases}$

10. $\begin{cases} y^2 + 4x = 2y + 11, \\ x + 4y = 14. \end{cases}$

ANNOUNCEMENTS

Text-Books on Mathematics